Classical communication and entanglement cost in preparing a class of multi-qubit states

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Abstract Recently, several similar protocols[J. Opt. B 4 (2002) 380; Phys. Lett. A 316 (2003) 159; Phys. Lett. A 355 (2006) 285; Phys. Lett. A 336 (2005) 317] for remotely preparing a class of multi-qubit states (i.e, $\alpha|0\cdots0\rangle+\beta|1\cdots1\rangle$) are proposed, respectively. In this paper, by applying the controlled-not (CNOT) gate, a new simple protocol is proposed for remotely preparing such class of states. Compared to the previous protocols, both classical communication cost and required quantum entanglement in our protocol are remarkably reduced. Moreover, the difficulty of identifying some quantum states in our protocol is also degraded. Hence our protocol is more economical and feasible.

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Quantum entanglement and classical communication are two elementary resources in quantum information field. With these two elementary resources and certain local unitary operations, lots of interesting and important works have been done[1-8], such as quantum teleportation[1], quantum secret sharing (QSS)[2], quantum key distribution(QKD)[3], etc. In 2000, a distinct application of quantum entanglement and classical communication, i.e., remote state preparation (RSP), was proposed by Lo[9]. In RSP by means of a prior shared entanglement and some classical communication, a pure known quantum state is prepared in a remote place via certain local unitary operations. Different from quantum teleportation, in RSP the sender Alice is assumed to completely know the state to be prepared remotely. Due to the prior knowledge about the quantum state, to some extend the classical communication and entanglement cost can be reduced in RSP process. For an example, Pati [10] has shown that for a qubit chosen from equatorial or polar great circles on a Bloch sphere, RSP requires only 1 forward classical bit from Alice to Bob, exactly half that of teleportation. On the other hand, RSP exhibits a stronger trade-off relation between the required entanglement and the classical communication cost than quantum teleportation. The RSP protocols proposed by Lo[9] and Bennett et al.[11] show that in the presence of a large amount of previously shared entanglement, the asymptotic classical communication cost of RSP for general states is one bit per qubit. However, for special states, RSP protocol is more economical than quantum teleportation.

So far, various RSP protocols have been put forward in many literatures[12-33]. Among these protocols, some[29-34] concentrate on the preparation of a class of multi-qubit state (i.e., $\alpha|0\cdots0\rangle+\beta|1\cdots1\rangle$). Specifically, in 2002, Shi et al.[29] proposed a scheme for remotely preparing a two-qubit entangled state

(i.e., $\alpha|00\rangle+\beta|11\rangle$) by consuming a three-qubit Greenberger-Horne-Zeilinger (GHZ) state and 1-bit classical communication; In 2003, by means of two pairs of entangled qubits and 2 classical bits, Liu et al.[30] proposed other scheme for preparing remotely a two-qubit entangled state (i.e., $\alpha|00\rangle+\beta|11\rangle$); Very recently, Dai et al.[31] proposed a RSP scheme of a four-qubit GHZ class state (i.e., $\alpha|0000\rangle+\beta|1111\rangle$) via two non-maximally entangled three-qubit GHZ states, and the necessary classical communication cost is 1 bit; Wang et al.[33] proposed a RSP protocol of a three-qubit state (i.e., $\alpha|000\rangle+\beta|111\rangle$) by taking both a three-qubit entangled state and a two-qubit entangled state as the quantum channel, and the necessary classical communication cost is 0.5 bits on average. Moreover, the clone protocol proposed by Zhan[34] is virtually a RSP protocol of a two-qubit entangled state (i.e., $\alpha|00\rangle+\beta|11\rangle$) with two Bell states and 2 classical bits. In this paper, we will propose a new simple protocol for remotely preparing a class of multi-qubit states (i.e., $\alpha|0\cdots0\rangle+\beta|1\cdots1\rangle$). Compared to these previous protocols[29-31, 33-34], both required quantum entanglement and classical communication cost in our this protocol are greatly reduced, as means that our protocol is more economical. This is a distinct advantage of our protocol and one will see it later.

For simplicity, let us firstly consider the remote preparation of a two-qubit entangled state (i.e., $\alpha|00\rangle+\beta|11\rangle$) with a Bell state as the quantum channel. Suppose Alice is the ministrant while Bob the state preparer. Alice wants to help Bob to prepare remotely a two-qubit entangled state $|P\rangle=\alpha|00\rangle+\beta|11\rangle$, where α is real and β is complex and $|\alpha|^2+|\beta|^2=1$. Bob dose not know the two coefficients of the state $|P\rangle$ but Alice dose. In addition, as mentioned before, the state taken as the quantum channel between Alice and Bob is a Bell state. Without loss of generality, we take it as

$$|\phi\rangle_{12}=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)_{12}. \tag{1}$$

Suppose qubit 1 belongs to Alice while qubit 2 to Bob. In order to help Bob to prepare the state $|P\rangle$, Alice performs a single-qubit projective measurement on her qubit 1 in a set of two mutually orthogonal basis vectors $\{|\psi\rangle,|\psi_\perp\rangle\}$, where

$$|\psi\rangle=\alpha|0\rangle+\beta|1\rangle, \qquad |\psi_\perp\rangle=\beta^*|0\rangle-\alpha|1\rangle. \tag{2}$$

Note that $|\psi\rangle$ is exactly the original state $|P\rangle$. These two mutually orthogonal basis vectors are related to the computation basis vectors $\{|0\rangle,|1\rangle\}$ and form a complete orthogonal basis set in a sing-qubit 2-dimensional Hilbert space. Then we observe the shared Bell state between Alice and Bob can be rewritten as

$$|\phi\rangle_{12} \;=\; \frac{1}{\sqrt{2}}[|\psi_\perp\rangle_1(\beta|0\rangle-\alpha|1\rangle)_2+|\psi\rangle_1(\alpha|0\rangle+\beta^*|1\rangle)_2]. \tag{3}$$

According to the equation (3), one can see that Alice's single-qubit projective measurement result should be $|\psi_\perp\rangle_1$ or $|\psi\rangle_1$. For each measurement result, its occurrence probability is 1/2. Incidentally, before Alice informs of Bob her measurement result, they agree that $|\psi_\perp\rangle_1$ corresponds to the classical bit "0".

If the measurement result is $|\psi_\perp\rangle_1$, Alice sends a classical bit "0" to Bob. After having received Alice's classical bit "0" in a certain interval, Bob knows that Alice's measurement result is $|\psi_\perp\rangle_1$ and the state of qubit 2 has collapsed to $(\beta|0\rangle-\alpha|1\rangle)_2$. Then to realize the preparation of the two-qubit entangled state $|P\rangle$, Bob carries out some actions as following.

Step (1) Bob performs the unitary operation $U_1=(|1\rangle\langle0|-|0\rangle\langle1|)_2$ on his qubit 2. After his this unitary operation, the collapsed state of qubit 2 $(\beta|0\rangle-\alpha|1\rangle)_2$ is transformed into

$$|T\rangle_2=(\alpha|0\rangle+\beta|1\rangle)_2. \tag{4}$$

Step (2) Bob introduces an auxiliary qubit 3 in the state $|0\rangle$. He performs a controlled-not (CNOT) gate operation $C_{2,3}$ by taking the qubit 2 as a control qubit and the qubit 3 as a target one. This

unitary operation transforms the joint state of the qubits 2 and 3 as following

$$C_{2,3}|T\rangle_2|0\rangle_3 = (\alpha|00\rangle + \beta|11\rangle)_{23}. \tag{5}$$

One can see that after the unitary operation the original state $|P\rangle$ has been successfully prepared in the qubits 2 and 3.

If Alice's measurement result is $|\psi\rangle_1$, the state of qubit 2 collapses to $(\alpha|0\rangle + \beta^*|1\rangle)_2$. Since Bob has no knowledge about the coefficients α and β, it seems that he can not convert this collapsed state to the state $\alpha|0\rangle + \beta|1\rangle$ via a certain unitary operation. This means Bob can not prepare the original state $|P\rangle$ by applying the CNOT gate. Apparently, Alice needs not to send any classical bit to Bob. However, as mentioned before, the coefficient α is assumed real while β complex in the beginning. Then one would like to ask whether the conversion can be realized in the case of the latter outcome provided that Alice lets Bob know some information about the coefficients. In fact, there really exist two special *cases*: (A) Both α and β are real; (B) α is $\frac{1}{\sqrt{2}}$ and β is $\frac{1}{\sqrt{2}}e^{i\theta}$ (θ is an arbitrary real parameter). In these two special cases, for the latter outcome, the preparation of the original state can also be realized at Bob's place. This can be seen from the following detailed analysis process. Incidentally, in the two special cases, two classical bits "10" or "11" instead of the single classical bit "0" are sent from Alice to Bob to realize his preparation. The first classical bit "1" corresponds to Alice's measurement result $|\psi\rangle_1$ and the second bit "0" ("1") to $Case(A)$ $((B))$.

$Case(A)$ Both α and β are real. After obtaining the measurement result $|\psi\rangle_1$, Alice sends Bob two classical bits "10" in a certain interval. According to these information, Bob knows that Alice's measurement result is $|\psi\rangle_1$ and both α and β are real. Then he can infer that the state of qubit 2 has collapsed to $(\alpha|0\rangle + \beta|1\rangle)_2$. In order to achieve his goal of preparing remotely the state $|P\rangle$, Bob repeats the step (2) proposed above. After this, Bob has successfully realized the preparation of the state $|P\rangle$ at his place. Therefore, in the case that both α and β are real, the original state $|P\rangle$ can always be deterministically prepared in the remote place.

$Case(B)$ α is $\frac{1}{\sqrt{2}}$ and β is $\frac{1}{\sqrt{2}}e^{i\theta}$ (θ is an arbitrary real parameter). If Alice gets $|\psi\rangle_1$, she sends two classical bits "11" to Bob. According to this information, Bob knows that the state of qubit 2 has collapsed to $\frac{1}{\sqrt{2}}(|0\rangle + e^{-i\theta}|1\rangle)_2$. To prepare the original state $|P\rangle$ in his place, Bob firstly performs the unitary operation $U_2 = (|0\rangle\langle 1| + |1\rangle\langle 0|)_2$, which transforms the above collapsed state into $e^{-i\theta} \times \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)_2$, and then he repeats the step (2). After this, the joint state of qubits 2 and 3 in his possession is

$$e^{-i\theta} \times \frac{1}{\sqrt{2}}(|00\rangle + e^{i\theta}|11\rangle)_{23}. \tag{6}$$

It is exactly the original state $|P\rangle$ except for an overall unimportant phase factor. This means for the special values of α and β, the remote preparation can also be deterministically realized.

By the above analyses, one can easily conclude that with one Bell state and one auxiliary qubit, the original state $|P\rangle$ can be prepared probabilistically or deterministically in a remote place. In a general case, the total success probability is $\frac{1}{2}$ (probabilistic) and the necessary classical communication cost is 0.5 ($\frac{1}{2} \times 1 + \frac{1}{2} \times 0 = 0.5$) forward bit **on average**. Nonetheless, if the state $|P\rangle$ belongs to the special states described in cases (A) and (B), the success probability of preparation can achieve 1 (deterministic), and the classical communication cost is increased to 1.5 ($\frac{1}{2} \times 1 + \frac{1}{2} \times 2 = 1.5$) bits **on average**. *It should be emphasized that the fraction of bit as the classical communication cost is not created first by this paper but has already occurred in Ref.[31].*

Now let us generalize the above protocol to a multi-qubit state case, i.e., the state to be prepared remotely is a $m(m \geq 3)$-qubit state

$$|P'\rangle = \alpha \prod_{i=1}^{m} |0\rangle_i + \beta \prod_{i=1}^{m} |1\rangle_i. \tag{7}$$

In this case, the demonstration of preparing the m-qubit state is very similar to the above process except for a little modification in the step (2). That is, after having got the collapsed state $(\alpha|0\rangle + \beta|1\rangle)_2$, instead of introducing one auxiliary qubit, Bob introduces m-1 auxiliary qubits each in the state $|0\rangle$. To achieve his goal of preparation, Bob performs the CNOT gate operations $C_{2,m+1}\cdots C_{2,4}C_{2,3}$ with the qubit 2 always as the control qubit and the other qubits as the target qubits. In this way, after having performed the local CNOT gate operations, the joint state $(\alpha|0\rangle + \beta|1\rangle)_2 \prod_{k=3}^{m+1}|0\rangle_k$ of the qubits of 2, 3, ..., m+1 becomes

$$\begin{aligned} & C_{2,m+1}\cdots C_{2,4}C_{2,3}(\alpha|0\rangle + \beta|1\rangle)_2 \prod_{k=3}^{m+1}|0\rangle_k \\ = \; & (\alpha|00...0\rangle + \beta|11...1\rangle)_{23...m+1} = \alpha\prod_{k=2}^{m+1}|0\rangle_k + \beta\prod_{k=2}^{m+1}|1\rangle_k. \end{aligned} \tag{8}$$

It is exactly the original state $|P'\rangle$ which needs to be prepared remotely. This means Bob has already successfully prepared the state $|P'\rangle$ at his place.

Similarly, in the process of preparing a multi-qubit state, Alice will get the measurement result $|\psi\rangle_1$ with probability $\frac{1}{2}$. In this case, due to the same reason mentioned previously, it seems that the original state $|P'\rangle$ cannot be prepared remotely and Alice needs not to send any classical bit to Bob. Likewise, there also exist two exceptions, i.e., the two coefficients of the state $|P'\rangle$ belongs to some special values as described in the $cases(A)$ and (B). In these two cases, applying the same method proposed previously, the remote preparation of the state $|P'\rangle$ can also be deterministically realized by applying the CNOT gate m-1 times.

Based on above analyses, one can easily see that, even in the case that the state to be prepared is a m-qubit ($m \geq 3$) state, by means of one Bell state and 0.5 classical bits on average, the preparation of the original state can also be realized probabilistically, and the success probability is at least $\frac{1}{2}$. The additional cost is that m-1 auxiliary qubits should be introduced and the local unitary operation of CNOT gate should be applied m-1 times. Similarly, if the state $|P'\rangle$ belongs to some special states, the remote preparation of the original state can be realized in a deterministic manner. The necessary classical communication cost is increased to 1.5 bits on average. Incidentally, it is obvious that the present RSP protocol of a class of multi-qubit state can be simply and straightforwardly generalized to the case that partial entangled state instead of Bell state is taken as quantum channel.

Table I: The comparisons between our protocol and the previous protocols. Q.S. denotes the quantum state to be prepared remotely in the protocol; S.Q.C. denotes the states taken as the quantum channel; C.C. denotes the necessary classical communication cost; I.D. denotes the entangled state needed to be identified; GHZS denotes the GHZ state; ES denotes the entangled state; and BS stands for the Bell state.

Protocol	Q.S.	S.Q.C.	C.C.	I.D.
Shi et al[29]	$\alpha\|00\rangle + \beta\|11\rangle$	one GHZS	1 bit	1-qubit state
Liu et al[30]	$\alpha\|00\rangle + \beta\|11\rangle$	two BSs	2 bits	2-qubit ES
Dai et al[31]	$\alpha\|0000\rangle + \beta\|1111\rangle$	two GHZSs	1 bit	2-qubit ES
Zhan et al[34]	$\alpha\|00\rangle + \beta\|11\rangle$	two BSs	2 bits	2-qubit ES
Wang et al[33]	$\alpha\|000\rangle + \beta\|111\rangle$	one GHZS and one BS	0.5 bit	2-qubit ES
Our protocol	$\alpha\prod_{i=1}^{m}\|0\rangle_i + \beta\prod_{i=1}^{m}\|1\rangle_i$	one BS	0.5 bit	1-qubit state

So far one can see that all the preparation goals in Refs.[29-31, 33-34] can also be achieved by using our this RSP protocol. Now let us make some detailed comparisons between our present protocol and the previous protocols Refs.[29-31,33-34]. Different factors of different protocols are summarized in

Table I. The following points can be abstracted via comparisons: ($p1$) In Refs.[29-31,33-34], the initial states taken as the quantum channel are some complicated states, such as two Bell states[30,34], one or more three-particle entangled states[29,31], or their combination[33]. In contrast, the required quantum entanglement as quantum channel in our protocol is only one Bell state. To our knowledge, so far preparation of five-photon entangled states has been achieved in experiment[37], however, preparation more-photon entanglement is still desired. Alternatively, when photon number is large, it is impossible to prepare multi-photon GHZ states according to the present-day technologies. Hence, in the case that m is large, the method in Ref.[29] is impossible in reality. Nevertheless, since the quantum state (Bell state) taken as quantum channel in our protocol is comparatively simple, the preparation and secure distribution difficulties of the initial state taken as quantum channel is greatly reduced. ($p2$) The amount of classical communication needed in Refs.[29-31,34] are much larger. For examples, with the same success probability of preparation, 1 classical bit is required by the protocols in Refs.[29,31] and two classical bits by the protocols in Refs.[30,34]. In contrast, in this protocol, the necessary classical communication cost is only 0.5 forward classical bits on average. Moreover, if the state $|P\rangle$ belongs to some special states, the preparation can be realized in a deterministically manner by consuming one extra classical bit on average. Hence compared to the protocols [29-31,34], our protocol consumes the minimum classical communication. ($p3$) In Refs.[30-31,33-34], Alice needs to complete the identification of two-particle entangled states. In this protocol, the identification Alice needs to complete is just a single-qubit state, disregarding completely how many qubits there are. Thus, it is obvious in our protocol the difficulty of Alice's identification on her quantum state is degraded. According to these comparisons, one can conclude that our protocol is much simple, economical and feasible.

To summarize, with one Bell state as the quantum channel, we have explicitly proposed a protocol for remotely preparing a class of multi-qubit state by applying the CNOT gate. Although the idea in the present scheme is simple, it can indeed realize the preparation of the class of m-qubit state. More important is that, compared to the previous protocols[29-31,33-34], our this protocol reduces greatly the required quantum entanglement and classical communication cost. Moreover, the difficulty of identifying quantum state can also be degraded. Hence our protocol is economical and feasible.

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